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Generalized Widder Theorem via fractional moments

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Abstract

We provide a necessary and sufficient condition for the representability of a function as the classical multidimensional Laplace transform, when the support of the representing measure is contained in some generalized semi-algebraic set. This is done by employing a method of Putinar and Vasilescu [M. Putinar, F.-H. Vasilescu, Solving moment problems by dimensional extension, Ann. of Math. (2) 149 (3) (1999) 1087–1107] for the corresponding multidimensional moment problem.

Keywords: Multidimensional fractional moment problem; Widder Theorem; Laplace transform

0. Introduction

A well-known theorem of Widder states that *a necessary and sufficient condition for a function $f:(0,\infty)\to\mathbb{R}$ to be representable in the form*

$$(\forall x>0)\quad f(x)=\int_{-\infty}^{\infty} e^{-xt}\,d\mu(t),$$

where μ is a positive measure over $\mathbb{R}$, is that $f(x)$ be continuous and of positive type (cf. [13, Ch. VI, §21]). This theorem has been generalized to the multidimensional case in several works,

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using various methods (see Akhiezer [1], Devinatz [3], Shucker [10] and their references, to mention a few), and has many applications. The same is true for closely related representation theorems, such as Bernstein's Theorem [2, Ch. 4] and the Paley–Wiener–Schwartz Theorem on the Fourier–Laplace transform [6, Ch. 7]. If, however, we wished to characterize those functions with a representing measure whose *support* is contained in some rather general, not necessarily convex, fixed set, these results would not be helpful.

On the other hand, Putinar and Vasilescu used in [9] a method of dimensional extension to solve the multidimensional moment problem. In their paper, the moment problem is translated to the problem of representation of a certain linear functional, which is obtained by means of the spectral theory of selfadjoint operators, over some special Hilbert space. The bonus in their method is that it enables them to characterize moment sequences, whose representing measure's support lies in a given semi-algebraic set.

The connection between a moment problem and the corresponding Laplace transform representation problem has been successfully established in the past (e.g. in [12], and the references therein). In this note we modify Putinar and Vasilescu's method of dimensional extension to obtain a generalized version of Widder's Theorem, which characterizes the functions that can be represented by the multidimensional Laplace transform of a measure with support in a given (generalized) semi-algebraic set. Essentially, non-negative integral powers of the variables are replaced by non-negative rational powers.

1. Preliminaries

Let $\mathcal{R}$ be an algebra of complex functions, such that $\overline{f} \in \mathcal{R}$ for all $f \in \mathcal{R}$ (that is, $\mathcal{R}$ is selfadjoint). We say that a linear functional Λ over $\mathcal{R}$ is *positive semi-definite* if $\Lambda(|f|^2) \geqslant 0$ for each $f \in \mathcal{R}$. When this is the case, one can define the semi-inner product $(f, g) := \Lambda(f\bar{g})$. Thus, if $\mathcal{N} = \{f \in \mathcal{R}\colon\ \Lambda(|f|^2) = 0\}$, then $\mathcal{R}/\mathcal{N}$ is an inner-product space. Hence, its completion, $\mathcal{H}$, is a complex Hilbert space. For simplicity, we often write r instead of $r + \mathcal{N}$ for elements $r \in \mathcal{R}/\mathcal{N}$.

The standard notations $\mathbb{R}_+ = [0, \infty)$, $\mathbb{Q}_+ = \mathbb{R}_+ \cap \mathbb{Q}$, etc. are used. Fix an $n \in \mathbb{N}$. For $t = (t_1, \ldots, t_n), \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}_+^n$, we write t^α for $t_1^{\alpha_1} \cdots t_n^{\alpha_n}$.

We let $\mathcal{P}_n$ denote the set of all complex polynomials with n real variables. By $\mathcal{Q}_n$ we shall denote the complex algebra of all "fractional polynomials" of positive rational exponents and n variables. That is, $\mathcal{Q}_n$ is the set of all of the functions in the form $\mathbb{R}_+^n \ni t \mapsto \sum_{\alpha \in \mathbb{Q}_+^n} a_\alpha t^\alpha$, where the a_α's are complex, and differ from zero only for a finite number of indices α.

Let $\mathbb{A}$ be a subsemigroup of $\mathbb{Q}_+^n$. A family of complex numbers $\delta = (\delta_\alpha)_{\mathbb{A}}$ induces the linear functional L_δ over the subalgebra of $\mathcal{Q}_n$ generated by $\{t^\alpha\colon\ \alpha \in \mathbb{A}\}$, defined by $L_\delta(t^\alpha) = \delta_\alpha$ for all $\alpha \in \mathbb{A}$. We say that δ is positive semi-definite if the functional L_δ is positive semi-definite.

For the rest of the section, $\mathcal{H}$ denotes an arbitrary complex Hilbert space.

Lemma 1.1. *Let A be a positive selfadjoint operator over $\mathcal{H}$, and let q_1, q_2 be positive real numbers. Then there exists a* unique *positive selfadjoint operator B, so that $B^{q_2} = A^{q_1}$, namely $B = A^{q_1/q_2}$.*

Proof. Let $E(\cdot; A)$ denote the resolution of the identity of the selfadjoint operator A. By [4, Theorem XII.2.9], a positive operator B satisfies the theorem's statement if and only if for every Borel set $\delta \subseteq \mathbb{R}_+$,

$$E\big(\delta^{1/q_2}; B\big) = E\big(\delta; B^{q_2}\big) = E\big(\delta; A^{q_1}\big) = E\big(\delta^{1/q_1}; A\big). \tag{1.1}$$

Since the mapping $\delta \mapsto \delta^{q_2}$ is bijective from the set of all Borel subsets of $\mathbb{R}_+$ into itself, (1.1) is equivalent to that

$$E(\delta; B) = E\big(\delta^{q_2/q_1}; A\big) \tag{1.2}$$

for all Borel sets $\delta \subseteq \mathbb{R}_+$. But by the same theorem from [4], there exists a unique positive selfadjoint operator B that satisfies (1.2), which is $B = A^{q_1/q_2}$. $\square$

We now state two results from [9].

Proposition 1.2. *(See [9, Proposition 2.1], originally from [7,8].) Let $T_1, \ldots, T_n$ be symmetric operators in $\mathcal{H}$. Assume that there exists a dense linear space $\mathcal{D} \subseteq \bigcap_{j,k=1}^n D(T_j T_k)$ such that $T_j T_k x = T_k T_j x$ for all $x \in \mathcal{D}$, $j \neq k$, $j, k = 1, \ldots, n$. If the operator $(T_1^2 + \cdots + T_n^2)_{|\mathcal{D}}$ is essentially selfadjoint, then the operators $T_1, \ldots, T_n$ are essentially selfadjoint, and their canonical closures $\overline{T_1}, \ldots, \overline{T_n}$ commute.*

Lemma 1.3. *(See [9, Lemma 2.2].) Let A be a positive densely defined operator in $\mathcal{H}$, such that $AD(A) \subseteq D(A)$. Suppose that $I + A$ is bijective on $D(A)$. Then A is essentially selfadjoint.*

2. Generalized Widder Theorem

Let $p = (p_1, \ldots, p_m)$, where p_k are real fractional polynomials in $\mathcal{Q}_n$. For this fixed set of polynomials, let $\theta_p : \mathbb{R}_+^n \to \mathbb{C}$ be defined as

$$\theta_p(t) := \big(1 + t_1^2 + \cdots + t_n^2 + p_1(t)^2 + \cdots + p_m(t)^2\big)^{-1}.$$

We denote by $\mathcal{R}$ the complex algebra generated by $\mathcal{Q}_n$ and the function θ_p.

The following is the main operator-theoretic result, leading to the moments theorem to follow.

Theorem 2.1. *Let Λ be a positive semi-definite functional over $\mathcal{R}$. Then there exists a unique representing measure for Λ. The support of that measure is contained in $\mathbb{R}_+^n$. Moreover, if $\Lambda(p_k|r|^2) \geqslant 0$ for all $r \in \mathcal{R}$, $1 \leqslant k \leqslant m$, then the support of that (unique) measure is a subset of $\bigcap_{k=1}^m p_k^{-1}(\mathbb{R}_+)$.*

Proof. Let $\mathcal{H}$ be the Hilbert space generated by Λ, as explained in Section 1. For $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m$, we define the operators T_i, P_j over $\mathcal{R}/\mathcal{N}$ by

$$T_i : r + \mathcal{N} \mapsto t_i r + \mathcal{N}, \qquad P_j : r + \mathcal{N} \mapsto p_j r + \mathcal{N}.$$

Let B be the operator $B := T_1^2 + \cdots + T_n^2 + P_1^2 + \cdots + P_m^2$. Then $B : \mathcal{R}/\mathcal{N} \to \mathcal{R}/\mathcal{N}$ is a positive operator, since for all $r \in \mathcal{R}$, $(Br, r) = \sum_{i=1}^n \Lambda(|t_i r|^2) + \sum_{j=1}^m \Lambda(|p_j r|^2) \geqslant 0$, by the positivity of Λ. Moreover, $I + B$ is bijective, since for all $r \in \mathcal{R}$, $(I + B)u = r$ for some $u \in \mathcal{R}$ if and only if $u = \theta_p r$. Therefore, by Lemma 1.3, B is essentially selfadjoint. Thus, by Proposition 1.2, the operators T_i and P_j are essentially selfadjoint for all $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m$. Moreover, the selfadjoint operators $A_1 := \overline{T_1}, \ldots, A_n := \overline{T_n}$ commute, and thus have a common resolution of the identity, E (cf. [11, Ch. IV, Theorem 10.3]). Set $A := (A_1, \ldots, A_n)$. For $r \in \mathcal{R}$, $r(A)$ will denote the normal operator $\int_{\mathbb{R}^n_+} r(t)E(dt)$, and for $q \in \mathbb{Q}^n_+$, A^q will stand for $f(A)$ where $f(t) = t^q$. We define the operators $T_i(q_i) : \mathcal{R}/\mathcal{N} \to \mathcal{R}/\mathcal{N}$, $1 \leqslant i \leqslant n$, by

$$T_i(q_i) : r + \mathcal{N} \mapsto t_i^{q_i} r + \mathcal{N},$$

and set $T(q) := T_1(q_1) \cdots T_n(q_n)$.

Claim 1. *For all $q \in \mathbb{Q}^n_+$, $T(q) \subseteq A^q$.*

To prove the claim, we first notice that $T(q)$ is positive for $q \in \mathbb{Q}^n_+$, as $\Lambda(t^q|r|^2) = \Lambda(|t^{q/2}r|^2) \geqslant 0$ for all $r \in \mathcal{R}$. Let $q = (\frac{k_1}{\ell_1}, \ldots, \frac{k_n}{\ell_n})$, where $k_1, \ldots, k_n \in \mathbb{N} \cup \{0\}$, $\ell_1, \ldots, \ell_n \in \mathbb{N}$. Fix an $1 \leqslant i \leqslant n$. Since the operator $T_i(\frac{1}{\ell_i})$ is positive, it has an (a priori, not necessarily unique) positive selfadjoint extension, $A_i(\frac{1}{\ell_i})$. Now, observe that $T_i = T_i(\frac{1}{\ell_i})^{\ell_i} \subseteq A_i(\frac{1}{\ell_i})^{\ell_i}$. But T_i is essentially selfadjoint and $A_i(\frac{1}{\ell_i})^{\ell_i}$ is selfadjoint, which implies that $A_i = \overline{T_i} = A_i(\frac{1}{\ell_i})^{\ell_i}$. Therefore, by the uniqueness part of Lemma 1.1, $T_i(\frac{1}{\ell_i}) \subseteq A_i(\frac{1}{\ell_i}) = A_i^{1/\ell_i}$. Hence, once again by Lemma 1.1, and the fact that

$$(\forall r_1, r_2 \in \mathcal{R}) \quad r_1(A)r_2(A) \subseteq (r_1r_2)(A) \tag{2.1}$$

(which follows readily from [11, Ch. IV, Theorem 10.3]),

$$T(q) = T_1\left(\frac{1}{\ell_1}\right)^{k_1} \cdots T_n\left(\frac{1}{\ell_n}\right)^{k_n} \subseteq \left(A_1^{1/\ell_1}\right)^{k_1} \cdots \left(A_n^{1/\ell_n}\right)^{k_n} = A_1^{k_1/\ell_1} \cdots A_n^{k_n/\ell_n} \subseteq A^q, \tag{2.2}$$

and the claim is proved.

Claim 2. *For all $r \in \mathcal{R}$,*

$$\Lambda(r) = \int_{\mathbb{R}^n_+} r(t)\big(E(dt)(1 + \mathcal{N}), 1 + \mathcal{N}\big). \tag{2.3}$$

In order to prove the claim, fix an $r \in \mathcal{R}$. Let the operator $r(T) : \mathcal{R}/\mathcal{N} \to \mathcal{R}/\mathcal{N}$ be the operator of multiplication by r. We shall show that $r(T) \subseteq r(A)$. By linearity and (2.1), it is sufficient to prove this for $r(t) = t^q$, $q \in \mathbb{Q}^n_+$, and for $r = \theta_p$. The first case is exactly Claim 1, since $r(T) = T(q)$ and $r(A) = A^q$. As for the case $r = \theta_p$, it follows from the fact that $\theta_p^{-1}(T) \subseteq \theta_p^{-1}(A)$, and so for all $f \in \mathcal{R}/\mathcal{N}$, $\theta_p(A)f = \theta_p(A)[\theta_p^{-1}(T)\theta_p(T)]f = \theta_p(A)\theta_p^{-1}(A)\theta_p(T)f = \theta_p(T)f$ (by (2.1)). Finally, to prove (2.3), we note that by the Spectral Theorem,

$$\begin{aligned}\Lambda(r) &= (r + \mathcal{N}, 1 + \mathcal{N}) = \big(r(T)(1 + \mathcal{N}), 1 + \mathcal{N}\big) = \big(r(A)(1 + \mathcal{N}), 1 + \mathcal{N}\big)\\ &= \int_{\mathbb{R}^n_+} r(t)\big(E(dt)(1 + \mathcal{N}), 1 + \mathcal{N}\big)\end{aligned}$$

(the domain of integration is $\mathbb{R}^n_+$ since the operators $A_1, \ldots, A_n$ are positive), and the claim is proved.

Consequently, the (positive) Borel measure μ over $\mathbb{R}^n$ defined by $\mu(\cdot) = (E(\cdot)(1+\mathcal{N}), 1+\mathcal{N})$ is a representing measure for Λ, whose support lies in $\mathbb{R}^n_+$. We have thus proved the existence part of the theorem.

The uniqueness of μ is proved as in [9], using an argument taken from [5]. Let us assume that there exists another positive measure ν over $\mathbb{R}^n_+$, that represents the functional Λ. It is clear that when this is the case, $\mathcal{R}/\mathcal{N}$ can be identified as a subspace of the Hilbert space $L^2(\nu)$. Hence, $\mathcal{H}$ can be identified as the closure of $\mathcal{R}/\mathcal{N}$ in $L^2(\nu)$. For all $1 \leqslant j \leqslant n$, let us now define the selfadjoint operators H_j over $L^2(\nu)$ by $H_j f := t_j f$. Denote the spectral measure of H_j by E_j. Since the operators $H_1, \dots, H_n$ commute, they have a joint spectral measure, E_H. Obviously, $T_j \subseteq H_j$ for all j. Since the operators H_j are closed, $A_j \subseteq H_j$ for all j. Therefore, $R(\zeta; A_j) \subseteq R(\zeta; H_j)$ for all $\zeta \in \mathbb{C} \backslash \mathbb{R}$, and so $R(\zeta; H_j)$ leaves $\mathcal{H}$ invariant, whence we conclude (cf. [4, Theorem XII.2.10]) that E_j also leaves $\mathcal{H}$ invariant for each $1 \leqslant j \leqslant n$. Thus, as $E_H(B_1 \times \cdots \times B_n) = E_1(B_1) \cdots E_n(B_n)$ for all Borel sets $B_1, \dots, B_n$ in $\mathbb{R}$, E_H leaves $\mathcal{H}$ invariant as well. In particular, for each Borel set B in $\mathbb{R}^n$, $I_B = E_H(B)1 \in \mathcal{H}$ (where I_B is the indicator function of B over $\mathbb{R}^n$). Since the simple functions are dense in $L^2(\nu)$, we infer that $\mathcal{H} = L^2(\nu)$, and so $A_j = H_j$ for each $1 \leqslant j \leqslant n$. In particular, $E = E_H$, and so for each Borel set B in $\mathbb{R}^n$, by the definition of μ,

$$\mu(B) = \big(E(B)(1+\mathcal{N}), 1+\mathcal{N}\big) = \big(E_H(B)1, 1\big) = \int_{\mathbb{R}^n} I_B \, d\nu = \nu(B),$$

and the proof of the uniqueness of μ is completed.

Assume now that $\Lambda(p_k|r|^2) \geqslant 0$ for all $r \in \mathcal{R}$ and $1 \leqslant k \leqslant m$. This condition is equivalent to the operators $P_1, \dots, P_m$ being positive. We recall that these operators are essentially selfadjoint. But for all such k, $P_k \subseteq p_k(A)$ by Claim 1, and $p_k(A)$ is selfadjoint; thus, $\overline{P_k} = p_k(A)$ is a positive selfadjoint operator. Equivalently, its spectral measure is supported by $\mathbb{R}_+$. But the spectral measure of $p_k(A)$ is $F_k(\delta) = E(p_k^{-1}(\delta))$. Hence, E itself is supported by $p_k^{-1}(\mathbb{R}_+)$. Since that is true for all $1 \leqslant k \leqslant m$, the support of E is therefore a subset of $\bigcap_{k=1}^m p_k^{-1}(\mathbb{R}_+)$. $\square$

Lemma 2.2. *Let $\vartheta \in \mathcal{P}_n$ be such that $\vartheta(t) > 0$ for all $t \in \mathbb{R}^n$, and let $p(t,s) \in \mathcal{P}_{n+1}$ ($t \in \mathbb{R}^n$, $s \in \mathbb{R}$) be such that $p(t, \vartheta^{-1}(t)) \equiv 0$. Then there exists a complex polynomial $q \in \mathcal{P}_{n+1}$ such that*

$$(\forall t,s) \quad p(t,s) = q(t,s) \cdot \big[s\vartheta(t) - 1\big].$$

Proof. This is a simple generalization of [9, Lemma 2.3]; simply replace their $\theta_{\mathbf{p}}$ by ϑ^{-1}. We omit the details. $\square$

Definition 2.3. Denote by $\widetilde{\mathcal{Q}}_n$ the complex algebra generated by $\mathcal{Q}_n$ and the algebra of all complex polynomials with one positive real variable. Its elements will take the form $p(t,s)$, $t \in \mathbb{R}^n_+$, $s \in \mathbb{R}_+$.

Proposition 2.4. *Let $\rho : \widetilde{\mathcal{Q}}_n \to \mathcal{R}$ be the mapping defined by $p(t,s) \mapsto p(t, \theta_p(t))$. Then ρ is a surjective algebras homomorphism, whose kernel is the ideal generated by the function*

$$\sigma(t,s) = s\theta_p(t)^{-1} - 1.$$

Proof. We first note that ρ is indeed well defined, since $\theta_p(t) \in \mathbb{R}_+$ for all $t \in \mathbb{R}_+^n$. It is clearly a surjective algebras homomorphism. Assume $p \in \ker(\rho)$, that is, $p(t, \theta_p(t)) = 0$ for all $t \in \mathbb{R}_+^n$. For $1 \leqslant j \leqslant n$, we let c_j denote the l.c.m. of all of the denominators of the exponents of t_j in the polynomial p. The mappings $u_j = t_j^{1/c_j}$ are bijective mappings from $\mathbb{R}_+$ onto itself. Replacing t_j by $u_j^{c_j}$ in the above equality yields

$$\left(\forall u \in \mathbb{R}_+^n\right) \quad p\left(u^c, \theta_p\left(u^c\right)\right) = 0. \tag{2.4}$$

The expression on the left side of (2.4), after the reduction of the fractions in the exponents of the u_j's, becomes a (not fractional) polynomial in $u = (u_1, \ldots, u_n)$. Hence, (2.4) is true (as equality of polynomials) for all $u \in \mathbb{R}^n$, and by Lemma 2.2, there exists a $q \in \mathcal{P}_{n+1}$ such that

$$\left(\forall u \in \mathbb{R}_+^n, s \in \mathbb{R}_+\right) \quad p\left(u^c, s\right) = q(u, s)\left[s\theta_p\left(u^c\right)^{-1} - 1\right].$$

We can now replace u by $t^{1/c}$, and by defining $\widetilde{q}(t, s) = q(t^{1/c}, s)$, we conclude that

$$p(t, s) = \widetilde{q}(t, s)\left[s\theta_p(t)^{-1} - 1\right]$$

for all $t \in \mathbb{R}_+^n$, $s \in \mathbb{R}_+$, where $\widetilde{q} \in \widetilde{\mathcal{Q}}_n$, as wanted. $\square$

Definition 2.5. Let $\gamma = (\gamma_\alpha)_{\alpha \in \mathbb{R}_+^n}$ be a family of non-negative numbers.

(1) We say that γ is continuous if the function $\alpha \mapsto \gamma_\alpha$ is continuous (as a function from $\mathbb{R}_+^n$ to $\mathbb{R}_+$).
(2) We say that γ is an (*n-dimensional*) *fractional moments family* if there exists a positive Borel measure μ over $\mathbb{R}_+^n$, such that

$$\left(\forall \alpha \in \mathbb{R}_+^n\right) \quad \gamma_\alpha = \int\limits_{\mathbb{R}_+^n} t^\alpha \, d\mu. \tag{2.5}$$

Note that (2.5) is equivalent to the (multidimensional) Laplace representation

$$\left(\forall \alpha \in \mathbb{R}_+^n\right) \quad \gamma_\alpha = \int\limits_{\mathbb{R}^n} e^{-\alpha \cdot s} \, d\nu(s)$$

obtained by the change of variable $t_i = e^{-s_i}$.

The following is the main theorem, whose proof is almost identical to that of Theorem 2.7 in [9], basing on our Theorem 2.1 and Proposition 2.4 instead of the parallel ones in [9], and using Lebesgue's Dominated Convergence Theorem to derive (2.5) for all of $\mathbb{R}_+^n$. For the sake of completeness, we include the details.

Theorem 2.6. *Let $\gamma = (\gamma_\alpha)_{\alpha \in \mathbb{R}_+^n}$ be a continuous family of non-negative numbers. Let $p_1, \ldots, p_m \in \mathcal{Q}_n$, $p_k(t) = \sum_{\xi \in I_k} a_{k\xi} t^\xi$ ($I_k \subseteq \mathbb{Q}_+^n$ is finite) for $k = 1, 2, \ldots, m$. Then γ is a fractional moments family with a representing measure whose support is a subset of $\bigcap_{k=1}^m p_k^{-1}(\mathbb{R}_+)$ if and only if there exists a positive semi-definite family*

$$\delta = (\delta_{(\alpha,\beta)})_{(\alpha,\beta)\in\mathbb{Q}_+^n\times\mathbb{Z}_+}$$

that satisfies:

(1) $\delta_{(\alpha,0)} = \gamma_\alpha$ *for all* $\alpha \in \mathbb{Q}_+^n$.
(2) $\delta_{(\alpha,\beta)} = \delta_{(\alpha,\beta+1)} + \sum_{j=1}^n \delta_{(\alpha+2e_j,\beta+1)} + \sum_{k=1}^m \sum_{\xi,\eta\in I_k} a_{k\xi}a_{k\eta}\delta_{(\alpha+\xi+\eta,\beta+1)}$ *for all* $(\alpha,\beta) \in \mathbb{Q}_+^n \times \mathbb{Z}_+$.
(3) *The families* $(\sum_{\xi\in I_k} a_{k\xi}\delta_{(\alpha+\xi,\beta)})_{(\alpha,\beta)\in\mathbb{Q}_+^n\times\mathbb{Z}_+}$ *are positive semi-definite for all* $k = 1,\dots,m$.

Moreover, the representing measure of γ (*with the properties mentioned above*) *is unique if and only if the family* δ *is unique.*

Proof. Necessity. Assume that γ is a fractional moments family with a representing measure μ, whose support is a subset of $E := \bigcap_{k=1}^m p_k^{-1}(\mathbb{R}_+)$. We define the family δ by

$$\big(\forall(\alpha,\beta)\in\mathbb{Q}_+^n\times\mathbb{Z}_+\big)\quad \delta_{(\alpha,\beta)} := \int_E t^\alpha\theta_p(t)^\beta\,d\mu. \tag{2.6}$$

Then δ is a positive semi-definite family, that satisfies (1). (2) is a result of the obvious equality

$$\int_E \big(\theta_p(t)\big(1+t_1^2+\dots+t_n^2+p_1(t)^2+\dots+p_m(t)^2\big)-1\big)t^\alpha\theta_p(t)^\beta\,d\mu = 0,$$

which is true for all $\alpha\in\mathbb{Q}_+^n$, $\beta\in\mathbb{Z}_+$. Finally, (3) is true since

$$\int_E p_k(t)\big|p\big(t,\theta_p(t)\big)\big|^2\,d\mu \geqslant 0$$

for all $p\in\widetilde{\mathcal{Q}}_n$, $1\leqslant k\leqslant m$.

Sufficiency. Let δ be as in the theorem's statement, and the algebra $\mathcal{R}$ be defined as in the beginning of this section. We define the linear functional Λ over $\mathcal{R}$ by

$$\Lambda(r) = L_\delta(p)$$

for all $r\in\mathcal{R}$, where L_δ is the linear functional induced by δ over $\widetilde{\mathcal{Q}}_n$, and $p\in\widetilde{\mathcal{Q}}_n$ is such that $r(t) = p(t,\theta_p(t))$ for all $t\in\mathbb{R}_+^n$. Λ is well defined, since by Proposition 2.4, $\mathcal{R}\cong\widetilde{\mathcal{Q}}_n/\mathcal{I}$, where $\mathcal{I}$ is the ideal in $\widetilde{\mathcal{Q}}_n$, generated by the element $s\theta_p(t)^{-1}-1$; and indeed, by (2), $(L_\delta)_{|\mathcal{I}} = 0$. Thus, Λ is a well-defined positive semi-definite mapping on $\mathcal{R}$. From (3) we deduce that $L_\delta(p_k|p|^2)\geqslant 0$ for all $p\in\widetilde{\mathcal{Q}}_n$, $1\leqslant k\leqslant m$, hence $\Lambda(p_k|r|^2)\geqslant 0$ for all $r\in\mathcal{R}$, $1\leqslant k\leqslant m$.

By virtue of Theorem 2.1, there exists a unique representing measure μ for Λ, whose support is a subset of E. Particularly, by (1),

$$\gamma_\alpha = \delta_{(\alpha,0)} = \int_E t^\alpha\,d\mu \tag{2.7}$$

for all $\alpha \in \mathbb{Q}_+^n$. But since the family γ is continuous, Lebesgue's Dominated Convergence Theorem implies that $\gamma_\alpha = \int_E t^\alpha \, d\mu$ for all $\alpha \in \mathbb{R}_+^n$, that is, γ is a fractional moments sequence, as wanted.

Assume that the family δ, that satisfies the conditions in the theorem's statement, is unique. Let μ_1, μ_2 be two representing measures for γ. By the uniqueness of δ, Eq. (2.6) and the discussion that follows, $\int_E t^\alpha \theta_p(t)^\beta \, d\mu_1 = \int_E t^\alpha \theta_p(t)^\beta \, d\mu_2$ for each $\alpha \in \mathbb{Q}_+^n$, $\beta \in \mathbb{Z}_+$. Therefore, $\int_E r \, d\mu_1 = \int_E r \, d\mu_2$ for all $r \in \mathcal{R}$, and by the uniqueness part of Theorem 2.1, it follows that $\mu_1 = \mu_2$.

Conversely, assume that μ is unique. Suppose that both δ_1, δ_2 satisfy the conditions in the theorem's statement. As explained above, δ_1, δ_2 induce the positive semi-definite linear functionals Λ_1, Λ_2, respectively, over $\mathcal{R}$, which, in turn, have the unique representing measures μ_1, μ_2, respectively (by Theorem 2.1). Both measures represent γ as a fractional moments family, and so, by the uniqueness of μ, $\mu_1 = \mu_2$, hence $\Lambda_1 = \Lambda_2$. Finally, for each $\alpha \in \mathbb{Q}_+^n$, $\beta \in \mathbb{Z}_+$, $(\delta_1)_{(\alpha,\beta)} = \Lambda_1(t^\alpha \theta_p(t)^\beta) = \Lambda_2(t^\alpha \theta_p(t)^\beta) = (\delta_2)_{(\alpha,\beta)}$, that is, $\delta_1 = \delta_2$. $\square$

Remark 2.7. Throughout this section, $\mathbb{Q}_+^n$ might have been replaced, e.g., by

$$\mathbb{A} := \left\{ \frac{k}{2^l} \colon k, l \in \mathbb{N} \cup \{0\} \right\}^n.$$

We are limited by the mere requirements that $\mathbb{A}$ be a subsemigroup of $\mathbb{Q}_+^n$ which contains $(\mathbb{N} \cup \{0\})^n$, and that $\frac{a}{2} \in \mathbb{A}$ for all $a \in \mathbb{A}$ (the latter is used in the proof of Claim 1 of Theorem 2.1). Such $\mathbb{A}$ is, of course, dense in $\mathbb{R}_+^n$.

As a concrete demonstration, we have the following immediate corollary of Theorem 2.6.

Corollary 2.8. *Denote $F := \{t \in \mathbb{R}_+^2 \colon t_1^2 \leqslant t_2\}$. In order for a continuous 2-dimensional family $(\gamma_\alpha)_{\alpha \in \mathbb{R}_+^2}$ of non-negative numbers to be representable in the form*

$$\gamma_\alpha = \int_F t^\alpha \, d\mu$$

where μ is a non-negative measure over F, it is necessary and sufficient that there exist a positive semi-definite family $(\delta_{(\alpha,\beta)})_{(\alpha,\beta) \in \mathbb{Q}_+^2 \times \mathbb{Z}_+}$, such that the following conditions hold:

(1) $\delta_{(\alpha,0)} = \gamma_\alpha$ *for all* $\alpha \in \mathbb{Q}_+^2$.
(2) $\delta_{(\alpha,\beta)} = \delta_{(\alpha,\beta+1)} + \delta_{(\alpha+2e_1,\beta+1)} + 2\delta_{(\alpha+2e_2,\beta+1)} + \delta_{(\alpha+4e_1,\beta+1)} - 2\delta_{(\alpha+2e_1+e_2,\beta+1)}$ *for all* $(\alpha, \beta) \in \mathbb{Q}_+^2 \times \mathbb{Z}_+$.
(3) *The family* $(\delta_{(\alpha+e_2,\beta)} - \delta_{(\alpha+2e_1,\beta)})_{(\alpha,\beta) \in \mathbb{Q}_+^2 \times \mathbb{Z}_+}$ *is positive semi-definite.*

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